Chaos in two dimensional CFTs.

A thesis presented to the faculty of
San Francisco State University
In partial fulfilment of
The Requirements for
The Degree

Master of Science
In
Physics

by

Gaurang Parkar

San Francisco, California

July 2017

CERTIFICATION OF APPROVAL

I certify that I have read *Chaos in two dimensional CFTs.* by Gaurang Parkar and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Science in Physics at San Francisco State University.

Kristan Jensen
Assistant Professor of Physics

Maarten Golterman
Professor of Physics

Jeff Greensite
Professor of Physics

Chaos in two dimensional CFTs.

Gaurang Parkar
San Francisco State University
2017

In this thesis we study properties of quantum chaos in context of AdS/CFT. We start off with a discussion of the origins of black hole physics and AdS/CFT, then discuss the relation between gravity and entanglement and quantum chaos through Mutual Information.

I certify that the Abstract is a correct representation of the content of this thesis.

08/03/17

Chair, Thesis Committee Date

ACKNOWLEDGMENTS

I would like to thank my parents for providing me this opportunity to study physics. I would also like to thank my thesis committee for spending their time and effort. A special thanks to Kristan Jensen for being a good adviser and providing me with helpful feedback.

TABLE OF CONTENTS

LIST OF FIGURES

Figure Page

Chapter 1

Overview

To understand quantum chaos in conformal field theories we must first understand some basics of conformal field theories and quantum gravity.

Chapter one gives an introduction of general relativity and black holes. We here learn the basic mathematics that is required to describe black holes, and also introduce Kruskal coordinate system and Kruskal and Penrose diagrams. Study of black holes is easier when we use Kruskal and Penrose diagrams.

Chapter two deals with black hole thermodynamics. These ideas were introduced by Stephen Hawking in the 70's, where several important theorems were proved which link black holes and thermal systems. Some of the explicit calculations that are done are the temperature of a Schwarzschild black holes, and thermodynamic quantities of BTZ black hole.

Chapter three is a discussion on AdS/CFT, and how string theory and a theory of quantum gravity can be linked to quantum field theories in one lower dimension.

Chapter 1

Overview

To understand quantum chaos in conformal field theories we must first understand some basics of conformal field theories and quantum gravity.

Chapter one gives an introduction of general relativity and black holes. We here learn the basic mathematics that is required to describe black holes, and also introduce Kruskal coordinate system and Kruskal and Penrose diagrams. Study of black holes is easier when we use Kruskal and Penrose diagrams.

Chapter two deals with black hole thermodynamics. These ideas were introduced by Stephen Hawking in the 70's, where several important theorems were proved which link black holes and thermal systems. Some of the explicit calculations that are done are the temperature of a Schwarzschild black holes, and thermodynamic quantities of BTZ black hole.

Chapter three is a discussion on AdS/CFT, and how string theory and a theory of quantum gravity can be linked to quantum field theories in one lower dimension.

Here we also discuss some properties of conformal field theory.

Chapter four is the main discussion. Starting of with the connection between entanglement and geometry, we discuss a method to calculate the entanglement entropy between quantum system. We also discuss quantum chaos through a quantity called mutual information.

Chapter 2

Black Holes

Black holes are regions in spacetime where the gravitational field is so strong that even light cannot escape. They were first predicted as solutions to Einstein's field equations by Schwarzschild. Black holes are formed in nature when a star collapses and its radius becomes so small that its escape velocity exceeds the speed of light.

In Newtonian gravity the escape velocity of a star is given by

$$v_e = \sqrt{\frac{2Gm}{r}} \tag{2.1}$$

where G is Newton's constant, m is the mass of the body and r is the radius of the body. So the star can form a black hole if

$$r < \frac{2Gm}{c^2} \tag{2.2}$$

The black hole has a horizon, and once anything falls into a black hole, it is trapped inside that region of spacetime forever.

This is a very naive description of a black hole. For a complete description of black holes we must go to the framework of General Relativity.

2.1 Special Relativity

General relativity was a theory of gravity developed by Albert Einstein in 1916. It describes gravity as a theory of spacetime itself! But first we discuss some concepts of special relativity.

In 4 dimensional flat spacetime (Minkowski space) the coordinate invariant distance is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.3}$$

we can define $x^0 = t$, $x^1 = x$, $x^2 = y$ and $x^3 = z$

$$ds^2 = -{dx^0}^2 + {dx^1}^2 + {dx^2}^2 + {dx^3}^2 \tag{2.4}$$

This equation can be rewritten as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.5}$$

Here the indices μ and ν run from 0 to 4. And η is the Minkowski metric.

Here s is the invariant distance of a point particle, so we can say that the action for a point particle must be proportional to the integral over s

$$S \propto \int ds$$

To match the units we must multiply with the mass of the particle

$$S = -m \int ds$$

$$S = -m \int \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu}$$

Let's parametrize this action by the proper time τ

$$S = -m \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau \tag{2.6}$$

to get equations of motion we set $\delta S = 0$. Doing this we are left with

$$\frac{d^2 x^\mu}{d\tau^2} = 0 \tag{2.7}$$

So we can see that in special relativity a particle moves in a straight line. We will see that particles do not move in a straight line in curved spacetime.

2.2 General Relativity

Einstein's theory of gravity says that matter curves space. So we must write down a coordinate invariant distance in curved spacetime. This can be done by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.8}$$

where $g_{\mu\nu}$ is called the metric tensor. Since we are not in flat space the metric tensor is not a constant but could depend on x^μ.

As in the case of Special Relativity we can write down the action for a particle in general relativity

$$S = -m \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau \tag{2.9}$$

Taking the variation of the action and setting it to zero we get

$$\frac{d^2 x^\beta}{d\tau^2} + \frac{1}{2} g^{\mu\beta} (\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}) \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{2.10}$$

defining Christoffel connection as

$$\Gamma^\beta_{\alpha\nu} = \frac{1}{2} g^{\mu\beta} (\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}) \tag{2.11}$$

we get

$$\frac{d^2 x^\beta}{d\tau^2} + \Gamma^\beta_{\alpha\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{2.12}$$

The above equation is known as *geodesic* equation and it describes the path a particle takes in a particular spacetime. So, once we have the metric tensor the path taken by a particle is completely defined.

Now we define the velocity vector as $v^\mu = \frac{dx^\mu}{d\tau}$ and then rewrite equation (2.12) as

$$\frac{dv^\beta}{d\tau} + \Gamma^\beta_{\alpha\nu} v^\alpha v^\nu = 0 \tag{2.13}$$

2.2.1 Einstein's Field Equations

Equation (2.12) gives us the equation of motion once we have the metric tensor, but we have not yet discussed how to obtain the metric tensor. The metric tensor is the solution of Einstein's Field Equations. These equations can be derived from varying the Einstein-Hilbert action and then setting the variation $\delta S = 0$.

The Einstein-Hilbert action with a cosmological constant is given by

$$S = \frac{1}{2\kappa^2} \int [(R - 2\Lambda) + \mathcal{L}_M] \sqrt{-g} d^4x \tag{2.14}$$

The field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.15}$$

where

- The constant $\kappa = 8\pi G$ is obtained by taking the Newtonian limit and comparing the equation to Newton's law of gravitation.

- $R_{\mu\nu}$ is the Ricci tensor obtained contracting the first and third indices of the Riemann Tensor $R^{\beta}_{\nu\rho\sigma}$

$$R^{\beta}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\beta}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\beta}_{\nu\rho} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\beta}_{\alpha\sigma} \tag{2.16}$$

and

$$R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} \tag{2.17}$$

- R is the scalar curvature given by contracting the Ricci Tensor.

$$R = g^{\mu\nu}R_{\mu\nu} \tag{2.18}$$

- Λ is the cosmological constant which acts as constant energy density in space-time and is also known as dark energy. The cosmological constant problem is one of the big unsolved problems in physics.

- $\mathcal{L}_M$ is the term in the Lagrangian density describing any matter fields appearing in the theory.

- $T^{\mu\nu}$ is the energy-momentum tensor of matter fields given by $\frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}$

2.2.2 Black Holes

Now that we have the machinery of general relativity we can start discussing black holes. Einstein's field equations are highly non-linear and so it is very hard to find solutions. One of the first solutions presented was by Schwarzschild in 1916. Consider the metric in spherical coordinates (r, θ, ϕ)

$$ds^2 = -(1 - \frac{2Gm}{r})dt^2 + \frac{1}{(1 - \frac{2Gm}{r})}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{2.19}$$

Here m is the mass of the black hole. This can be verified by going to the Newtonian approximation and then applying Newton's law of gravitation. This equation satisfies Einstein's equations for empty space with no cosmological constant.

We can see that as $r \to \infty$ we approach the metric of flat spacetime. We can also see that there is a singularity at $r = r_0 = 2Gm$ as the denominator of the second term on the right hand side goes to zero. Initially this was thought to be the minimum radius of a body of mass m but later it was understood that this is just a coordinate singularity, which can be better understood by doing a coordinate transformation.

2.2.3 Kruskal coordinates

Kruskal coordinates are defined by replacing t and r by a new time coordinate u and a new spacial coordinate v.

For the 'interior' region $0 < r < r_0$

$$u = \left(\frac{r}{r_0} - 1\right)^{1/2} e^{r/2r_0} \sinh\left(\frac{t}{2r_0}\right) \tag{2.20}$$

$$v = \left(\frac{r}{r_0} - 1\right)^{1/2} e^{r/2r_0} \cosh\left(\frac{t}{2r_0}\right) \tag{2.21}$$

and for the 'exterior' region $r > r_0$

$$u = \left(1 - \frac{r}{r_0}\right)^{1/2} e^{r/2r_0} \sinh\left(\frac{t}{2r_0}\right) \tag{2.22}$$

$$v = \left(1 - \frac{r}{r_0}\right)^{1/2} e^{r/2r_0} \cosh\left(\frac{t}{2r_0}\right) \tag{2.23}$$

In these new coordinates the Schwarzschild metric takes the form

$$ds^2 = \frac{4r_0}{r} e^{-r/r_0}(-du^2 + dv^2) + r^2 d\Omega^2 \tag{2.24}$$

where $\Omega^2 \equiv d\theta^2 + sin^2\theta d\phi^2$

For simplicity of analysis we shall consider only the radial motion so we can set $d\Omega = 0$ for our analysis. After doing these coordinate transformations we now talk about some of the properties of the Schwarzschild spacetime (see figure 2.1).

- The solution depends on r, but r should be regarded as a function of u and v

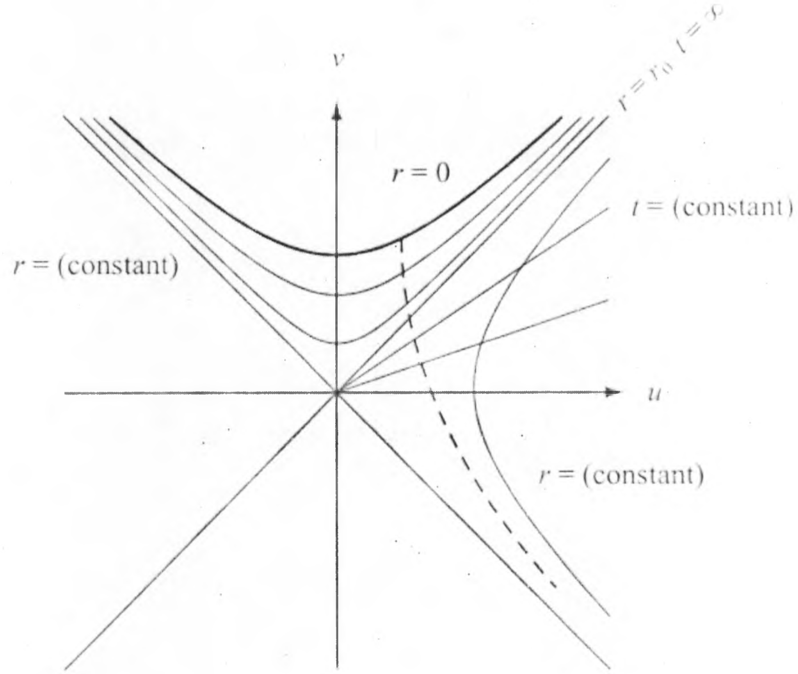

Figure 2.1: Kruskal diagram for Schwarzschild metric. Light travels at 45°. Once a particle passes the horizon, it reaches the singularity. Image taken from [8].

obtained from equations (2.19) and (2.20) or equations (2.21) and (2.22)

$$\left(1 - \frac{r}{r_0}\right) e^{r/r_0} = u^2 - v^2 \tag{2.25}$$

- There is no singularity at $r = r_0$ but the singularity at $r = 0$ still exists. This tells us that there is a *real* singularity at $r = 0$.

- We can discuss the path that light travels by setting $ds = 0$, we find that light travels at paths where $du = \pm dv$, this gives us $u = \pm v + c$. So we can say that light travels at 45° lines in this diagram.

- Surface of constant r is given by hyperbola as shown in equation (2.25).

- A path taken by the particle is shown by a dotted line in the figure. One can easily see that once a particle crosses the horizon, it will hit the singularity unless it travels faster than the speed of light.

For a better discussion Kruskal coordinates one can refer to [?]). We will also discuss more about Kruskal coordinates when we discuss other spacetime.

A lot more can be said about general relativity and black holes. For more details one can refer to textbooks of general relativity like see [3] and [15]). But we are going to end the discussion here and talk about an interesting phenomena of black hole thermodynamics in the next chapter.

Chapter 3

Black Hole Thermodynamics

In this chapter we will discuss that black holes that we have studied in the previous chapter behave like thermodynamic systems. These results were published by Hawking and Beckenstien in the 70s.

3.1 Thermodynamics and Statistical Mechanics

We will use canonical and microcanonical ensemble for the study of statistical mechanics and derive the laws of thermodynamics using them. In microcanonical ensemble we

3.2 An Analogy

In chapter 2 we discussed properties of a Schwarzschild black hole. The horizon of the black hole is $r_0 = 2GM$. So the area is given by

$$A = 4\pi r_0^2 = 16\pi G^2 M^2 \tag{3.1}$$

Classically nothing comes out of a black hole, this suggests that the area never decreases. It was later proved that the area of the black hole always increases [?]. The second law of thermodynamics says that entropy always increases, so using the analogy we can naively say that the area of a black hole is related to the entropy.

A black hole has only a few independent parameters like mass, charge and angular momentum. These results are known as 'no hair' theorems (see [2]). So once a black hole is formed, it's properties are independent of anything that formed it. To describe a thermodynamic system it is not necessary to know everything about all molecules but only certain parameters like pressure, temperature, etc. Since a black hole is described only by a few parameters we can say that it must behave like thermodynamic systems.

In the study of statistical mechanics we describe the system by microstates, but in without knowing a quantum theory for gravity it is not clear that what these micro-states in black holes are. Another striking feature of this analogy is

- We have postulated that the entropy of a black hole is proportional to its area.
- In statistical mechanics the entropy of a system is proportional to its volume.

This leads us to postulate that a black hole in d dimensions must describe a quantum system in one lower dimension. This observation is useful and has been realized in the AdS/ CFT correspondence. AdS/CFT correspondence states that a quantum field theory in d dimensions is 'dual' to a theory of gravity in $d+1$ dimensions. We will discuss more about AdS/CFT correspondence in the next chapter.

3.3 Hawking Radiation

In the 70's Steven Hawking and Jacob Bekenstein proved results that stated that black holes have a temperature and hence must radiate energy. The corresponding temperature is called Hawking Temperature. Hawking computed this by quantizing matter fields in black hole background, but we will use a simple way to derive this using smoothness of Euclidean spacetime.

3.3.1 Path Integral in Euclidean Spacetime

Although we do not have a quantum theory for gravity we can attempt to create a path integral definition of quantum gravity. One can define the following path integral

$$Z = \int \mathcal{D}[g, \phi] e^{iS_L[g,\phi]} \tag{3.2}$$

Here S_L is the gravitational action as a function of the metric g. One can do a 'Wick rotation' of the time axis by 90° but substituting $t = -i\tau$. So the path integral becomes

$$Z = \int \mathcal{D}[g, \phi] e^{-S_E[g,\phi]} \tag{3.3}$$

Here the S_E is the Euclidean action $S_E = -iS_L$ and is real for real fields.

The probability amplitude to go from a configuration (g_1, ϕ_1, t_1) to the configuration (g_2, ϕ_2, t_2) is given by

$$\langle (g_2, \phi_2), t_2 | (g_1, \phi_1), t_1 \rangle = \int \mathcal{D}[g, \phi] e^{iS_L[g,\phi]}. \tag{3.4}$$

In the Schrodinger picture we can express the same quantity as

$$\langle (g_2, \phi_2) | e^{iHt_2} e^{iHt_1} (g_1, \phi_1) \rangle = \langle (g_2, \phi_2) | e^{iH(t_2 - t1)} (g_1, \phi_1) \rangle \tag{3.5}$$

We now assume that $(g_1, \phi_1) = (g_2, \phi_2)$, and write $t_2 - t_1 = -i\beta$ (here β is the inverse temperature in units where the Boltzman constant (k_b) is one) and sum over the complete set of eigenstates (ψ_n, E_n) of the Hamiltonian we get

$$Z = \sum_n e^{-\beta E_n} \tag{3.6}$$

Equation (3.6) represents the same system as in equation (3.3) where the fields (g, ϕ) are periodic in τ and with the definition $t_2 - t_1 = -i\beta$, the period of τ is β.

Imposing periodicity of τ let's us compute the temperature.

3.4 Temperature of Schwarzschild Black Hole

Now that we have used an analogy to convert the path integral of a field theory to a partition function of a thermodynamic system, we can calculate the temperature of a black hole. Consider Schwarzschild metric from equation (2.19)

$$ds^2 = -Vdt^2 + V^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin\theta d\phi^2 \tag{3.7}$$

where

$$V = \left(1 - \frac{2GM}{r}\right) \tag{3.8}$$

as discussed earlier we continue the solution to Euclidean signature by setting $t = -i\tau$, with the period of τ being β. Thus we have,

$$ds^2 = Vd\tau^2 + V^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{3.9}$$

We have $V = 0$ for $r = 2GM$, thus we can expand V in a Taylor series about $r = r_0 = 2GM$. So up to $O(r - r_0)$

$$\begin{aligned} V &= V|_{r=r_0} + V'|_{r=r_0}(r - r_0) \\ &= V'|_{r=r_0}(r - r_0) \end{aligned} \tag{3.10}$$

Thus we have

$$ds^2 = \frac{1}{V'}\left(V'^2(r-r_0)d\tau^2 + \frac{dr^2}{r-r_0}\right) + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{3.11}$$

Now let $r - r_0 = \epsilon^2$, so $dr = 2\epsilon d\epsilon$. Using these substitutions equation (3.11) becomes

$$\begin{aligned} ds^2 &= \frac{1}{V'}\left(\epsilon^2 V'^2 d\tau^2 + 4d\epsilon^2\right) + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \\ &= \frac{4}{V'}\left(\epsilon^2\left(\frac{V'}{2}d\tau\right)^2 + d\epsilon^2\right) + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \end{aligned} \tag{3.12}$$

Now, we need to ensure that there is no conical singularity at $\epsilon \to 0$. This can be done by imposing that the ratio of circumference (going around in τ) to the radius (going around in ϵ) at $\epsilon \to 0$ is 2π. This fixes the periodicity of τ.

$$\begin{aligned} \frac{V'}{2}\tau &\sim \frac{V'}{2}\tau + 2\pi \\ \tau &\sim \tau + \frac{4\pi}{V'} \end{aligned} \tag{3.13}$$

Now, the period of τ is β. Thus we must have

$$\begin{aligned} \frac{4\pi}{V'}|_{r=r_0} &= \beta \\ T &= \frac{1}{8\pi GM} \end{aligned} \tag{3.14}$$

If we restore factors of $\hbar$, c and k_b the temperature of a black hole is

$$T = \frac{\hbar c^3}{8\pi G M k_b} \tag{3.15}$$

If we calculate the temperature of a solar mass black hole in SI units we get

$$T_{\odot} = 61.78 \times 10^{-9} K \tag{3.16}$$

So we can see that black holes are incredibly cold and so the radiation emitted by them is very low.

3.4.1 Saddle-Point Approximation of the Partition Function

Now that we have calculated the temperature we can use an approximation to calculate other thermodynamic quantities like the average energy and the entropy of the system.

We know that it is hard to compute the full path integral for this theory but we can say go in the semi-classical regime and say that the dominant contribution to the path comes from the extremum of the action (saddle point approximation), so under this approximation

$$Z \approx e^{-\bar{S}_E} \equiv e^{-\beta W} \tag{3.17}$$

where $\bar{S}_E$ is the classical solution to the action. And W is defined to be the effective thermodynamic potential.

$$W = E - TS \tag{3.18}$$

where T is the temperature and S is the entropy. With the help of this approximation we can calculate useful information for this system.

The average energy can be defined as

$$\langle E \rangle = \frac{1}{Z}\sum_n E_n e^{-\beta E_n} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{\partial \log Z}{\partial \beta} = \frac{\partial \bar{S}_E}{\partial \beta} \tag{3.19}$$

The probability p_n for being in the nth state is given by

$$p_n = \frac{1}{Z} e^{-\beta E_n} \tag{3.20}$$

Using these results we can calculate a formula for the entropy

$$S = -\sum_n p_n \log p_n \tag{3.21}$$

from equations (3.8), (3.9) and (3.10) we can say

$$\begin{aligned} S &= -\frac{1}{Z}\sum_n e^{-\beta E_n} \log \frac{e^{-\beta E_n}}{Z} \\ &= -\frac{1}{Z}\sum_n e^{-\beta E_n}(-\beta E_n - \log Z) \\ &= \beta \frac{\partial \bar{S}_E}{\partial \beta} + \log Z = \beta \langle E \rangle + \log Z \end{aligned} \tag{3.22}$$

These equations can now be used to calculate useful thermodynamic properties of black holes.

3.5 Some comments

- We have calculated the temperature of black holes but not other thermodynamic quantities like entropy, we need to know the explicit form of the action and the boundary terms of the action. Calculations of thermodynamic quantities for a special type of black hole the BTZ black hole are done in Appendix A. We cam calculate the entropy of the black hole using this approach and compare it with thel Beckinstein-Hawking formula:

$$S = A/(4G_N) \tag{3.23}$$

- We have described a black hole as thermodynamic systems. In statistical mechanics thermodynamic systems have certain 'microstates'. But without a quantum theory for gravity, we cannot have a description of these 'microstates' in a black hole.

- Black holes radiate energy through the Hawking radiation, but the information of the 'things' that the black holes is made of is lost. No matter what the black hole is made of, the same radiation comes out. This is the famous black hole information paradox.

Chapter 4

AdS/CFT

AdS/CFT correspondence was born out of superstring theory. It claims that certain quantum field theories are equivalent to certain string theories in one higher dimension. String theories are theories of gravity, so this correspondence has helped link theories without gravity to theories with gravity. We are not going to discuss string theory here, (for more references see (... [9] and [16])) as knowledge of string theory is not essential in the study of most features of AdS/CFT. AdS/CFT claims duality between two theories:

Strongly-coupled 4-dimensional gauge theory = Gravitational theory in 5-dimensional AdS spacetime.

A gauge theory describes all the forces we see in nature except gravity. For example Maxwell theory of electromagnetism is a U(1) gauge theory. When a gauge theory is strongly coupled it is often hard to analyze it as we have to consider loop corrections to higher orders. AdS/CFT claims that we can understand strongly coupled gauge theories using AdS spacetime.

The relation that we have mentioned above corresponds to the case with zero temperature. At finite temperature, we must replace the AdS spacetime with an AdS black hole. A strongly coupled gauge theory at finite temperature is dual to Gravitational theory in AdS black hole. Thus the study of black holes was necessary in the study of AdS/CFT correspondence.

Susskind in[13] suggested that a theory of gravity in d dimensions can be described by a theory with no gravity in $d-1$ dimensions, this was called the holographic principle. AdS/CFT is a realization of this idea. The correspondence is still a conjecture and has not yet been proven, but there have been more than 10000 papers on it and it seems to be consistent with all calculations. We are not going to discuss the origin of AdS/CFT here, but assume that the correspondence exists and do other discuss some features that come out of it. I am grateful to Makoto Natsuume as most of the discussion in this chapter is from [8].

4.1 Curved Spaces

Anti De-Sitter spacetime is a hyperbolic spacetime with constant negative curvature. But before we study hyperbolic spaces we will first study curved Euclidean spaces. Consider the metric in three dimensions.

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{4.1}$$

We can now constrain ourselves to a sphere of radius l in this space by imposing:

$$x^2 + y^2 + z^2 = K^2 \tag{4.2}$$

We can now switch to spherical coordinates by

$$\begin{aligned} x &= l \sin\theta \cos\varphi \\ y &= l \sin\theta \sin\varphi \\ z &= l \cos\theta \end{aligned} \tag{4.3}$$

The metric now looks like

$$ds^2 = l^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.4}$$

This metric has constant positive Ricci Scalar (positive curvature) given by

$$R = \frac{2}{l^2} \tag{4.5}$$

Anti De-Sitter space is a hyperbolic space. It is difficult to visualize hyperbolic space as it cannot be embedded inside Euclidean space, but it can be embedded in Minkowski space. Consider the metric

$$ds^2 = -dx^2 + dy^2 + dz^2 \tag{4.6}$$

with the constraint

$$-x^2 + y^2 + z^2 = -l^2 \tag{4.7}$$

To solve for these constraints we can make the following coordinate transformations

$$\begin{aligned} x &= l \sinh \rho \cos \varphi \\ y &= l \sinh \rho \sin \varphi \\ z &= l \cosh \rho \end{aligned} \tag{4.8}$$

the result is

$$ds^2 = l^2(d\rho^2 + \sinh^2 \rho d\varphi^2) \tag{4.9}$$

The above metric is a metric for hyperbolic space. We can see that it does not naturally have a time-like direction. This is hyperbolic space not hyperbolic spacetime. This space has a negative Ricci scalar given by

$$R = -\frac{2}{l^2} \tag{4.10}$$

4.2 The AdS Spacetime

Now that we have discussed spaces, lets talk about spacetime. The AdS_2 spacetime can be embedded in three dimensional Minkowski spacetime with two time-like dimensions. Consider

$$ds^2 = -dz^2 - dy^2 + dx^2 \tag{4.11}$$

and the constraint

$$-z^2 - y^2 + x^2 = -l^2 \tag{4.12}$$

This can be solved by doing the following coordinate transformations

$$\begin{aligned} z &= l\cosh\rho\cos\bar{t} \\ y &= l\cosh\rho\sin\bar{t} \\ x &= l\sinh\rho \end{aligned} \tag{4.13}$$

The metric becomes,

$$ds^2 = l^2(-\cosh^2\rho d\bar{t}^2 + d\rho^2) \tag{4.14}$$

These coordinates are called global coordinates. We have embedded AdS into flat spacetime with two timelike dimensions z and y, the Ads has only one timelike dimension.

The time coordinate $\bar{t}$ has periodicity 2π (from equation 4.14). Also note that the AdS spacetime solves Einstein's field equations with constant negative curvature $R = -2/l^2$.

4.2.1 Various coordinate systems

In the last section we have discussed global AdS spacetime using coordinates $(\rho, \bar{t})$. We can also use different coordinates system when required.

Static Coordinates

This system of coordinates is defined by defining a coordinate $\bar{r} \equiv \sinh \rho$. The metric becomes

$$ds^2 = l^2 \left(-(\bar{r}^2 + 1) d\bar{t}^2 + \frac{\bar{r}^2}{\bar{r}^2 + 1} \right) \tag{4.15}$$

... include figure

Poincare coordinates

This system of coordinates is the most often used system. It is defined by making coordinate transformations in equations (4.11) and (4.12) as

$$\begin{aligned} z &= \frac{lr}{2} \left(-t^2 + \frac{1}{r^2} + 1 \right) \\ y &= lrt \\ x &= \frac{lr}{2} \left(-t^2 + \frac{1}{r^2} - 1 \right) \end{aligned} \tag{4.16}$$

here $r > 0$. The metric becomes

$$ds^2 = l^2 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) \tag{4.17}$$

This system is also useful to compare with the AdS black hole. By substituting $z \equiv 1/r$ the metric can also be written as

$$ds^2 = \frac{l^2}{z^2}(-dt^2 + dz^2) \tag{4.18}$$

This can also be extended to higher dimensions

$$ds^2 = \frac{l^2}{z^2}(-dt^2 + dz^2 + (dx^i)^2) \tag{4.19}$$

4.3 Conformal Field Theory

We have discussed the duality between conformal field theory and anti-desitter space, so let's discuss conformal field theory in this section.

Scale Invariant Gauge Theory (Classical)

For field theories that are scale invariant, we expect that physics does not change under the transformation

$$x^\mu \to a x^\mu \tag{4.20}$$

This is an important property of four-dimensional gauge theories, for example Maxwell Theory.

Scale Invariant Gauge Theory (Quantum Theory)

Gauge theories discussed previously are gauge invariant classically in four dimensions. However, they are not scale invariant quantum mechanically. Renormalization introduces a renormalization scale which breaks the scale invariance of these gauge theories.

However, there is a special class of gauge theories that are scale invariant even quantum mechanically. We will only focus on such theories. In particular, the $\mathcal{N} = 4$ super-Yang-Mills theory has these symmetries.

Consequences of Scale Invariance

In this section we will see how scale and Poincare invariance in four-dimensions constrains five-dimensional spacetime. Consider a five dimensional spacetime.

$$ds_5^2 = \Omega(\omega)^2(-dt^2 + (dx^i)^2) + d\omega^2 \tag{4.21}$$

This metric has Poincare invariance in four dimensions. We to now determine the factor $\Omega(\omega)$ using scale invariance. The metric should be invariant under the transformation $x^\mu \to ax^\mu$, this implies $\Omega(\omega) \to a^{-1}\Omega(\omega)$. This tells us that ω must transform non trivially. The line element along ω is $d\omega^2$, so for the theory to be invariant only translations in ω are allowed. We can write the translation as

$$\omega \to \omega + l \log a \tag{4.22}$$

where l is some length scale. This transformation and scale invariance can now uniquely determine $\Omega(\omega)$. Under these transformations the metric becomes

$$ds_5^2 = e^{-2\omega/l}(-dt^2 + (dx^i)^2) + d\omega^2 \tag{4.23}$$

we can now define $r = e^{-2\omega/l}$, so

$$ds_5^2 = \left(\frac{r}{l}\right)^2 (-dt^2 + (dx^i)^2) + \left(\frac{l}{r}\right)^2 dr^2 \tag{4.24}$$

Which gives us precisely the AdS metric. The length scale l is the AdS radius.

4.4 AdS/CFT

Now, the AdS/CFT correspondence says that

$$Z_{CFT} = Z_{AdS_5} \tag{4.25}$$

The left hand side is the partition function of a gauge theory with scale invariance and the right hand side is the partition function of string theory in AdS_5. This relation is called GKP-Witten relation. We will make use of this duality in the next chapter.

Chapter 5

Black Holes and Chaos

....

5.1 Entanglement and Gravity

Entanglement is a central property of any quantum system. It plays an important role in describing many body-systems, quantum field theories,etc. We can write the state of any pure system as

$$|\Phi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle \tag{5.1}$$

where A and B are subsystems of the system. A system is defined to be entangled if it cannot be defined as product states in equation (5.1). It turns out that all quantum systems cannot be described by equation (5.1).

Given some basis $|\phi_n\rangle$ for A and $|\phi_m\rangle$ for B we can write down a general state for the system.

$$|\Phi\rangle = \sum_{m,n} c_{m,n} |\phi_n\rangle \otimes |\phi_m\rangle \tag{5.2}$$

where $c_{n,m}$ are complex coefficients defined by the normalization condition

$$\sum_{n,m} |c_{m,n}|^2 = 1 \tag{5.3}$$

We can now define a density matrix operator for subsystem A with orthogonal states $|\phi_A^i\rangle$ and associated probability p_i.

$$\rho_A = \sum_{n} p_i |\phi_A^i\rangle \langle\phi_A^i| \tag{5.4}$$

The density matrix is a Hermitian matrix with unit trace and non-negative eigenvalues p_i.

We can compute the expectation value of an observable using the density matrix.

$$\langle \mathcal{O}_A \rangle = tr(\rho \mathcal{O}_A) \tag{5.5}$$

Starting from a state $|\Phi\rangle$ it is possible to determine the density matrix of a subsystem and thus the associated ensemble (for an explicit calculation see [14]).

We can now define a quantity σ by

$$\sigma = -tr(\rho \log \rho) \tag{5.6}$$

We argue that σ can be regarded as a measure of disorder in a system. A pure ensemble has maximum order as we can verify that $\sigma = 0$ for pure ensemble, and in a random ensemble $\sigma = \log N$ where N is the number of states (....see page 189 of [11]).

In a thermodynamic system the entropy is regarded as the degree of disorder. So it turns out that the quantity σ is related to the entropy of the system. So we can define the entropy as

$$\begin{aligned} S &= \sigma \\ S &= -tr(\rho \log \rho) \end{aligned} \tag{5.7}$$

This is known as Von Neumann entropy named after John Von Neumann. It is also known as entanglement entropy.

We have seen earlier that starting with a state $|\Phi\rangle$ it is possible to determine the density matrix of the subsystem. Let's now consider the reverse question: given ρ_A for a quantum system is it possible to find a pure state of some larger system such that ρ_A is the reduced density matrix of the subsystem A? This process is called purification. There are infinite number of such purifications in general. For an ensemble $\rho(|\phi_A^i\rangle, p_i)$ in Hilbert space $\mathcal{H}_A$, we can describe a general purification as

$$|\Phi\rangle = \sum_i \sqrt{p_i}\, |\phi_A^i\rangle \otimes |\phi_B^i\rangle \tag{5.8}$$

where $|\phi_B^i\rangle$ are orthogonal states in Hilbert space $\mathcal{H}_B$. This representation is called

Schmidt decomposition and it is possible to represent any state of a combined system in this way.

One example is the idea that a thermal state (canonical ensemble) can be considered a system weakly coupled to a larger system called heat bath. The full system including the heat bath can be considered as a pure state and the entropy can of the ensemble can be understood as entanglement with the bath.

Sometimes it is useful to consider a simplified purification by choosing the purifying system as a copy (here the original system is L and we introduce an identical system R) of the original system and then considering the state:

$$|\Phi\rangle = \frac{1}{Z^{1/2}} \sum_i e^{-\beta E_i/2} |E_i\rangle_R \otimes |E_i\rangle_L \tag{5.9}$$

this state is know as the Thermofield Double state.

We can now find the density matrix for the L system by taking a partial trace over R.

$$\rho_L = \frac{1}{Z} \sum_i e^{-\beta E_i} |E_i\rangle \langle E_i| \tag{5.10}$$

5.2 Two sided Black holes

These ideas of entanglement have some useful applications in the AdS/ CFT correspondence. In the previous chapter we had mentioned that a black hole in AdS

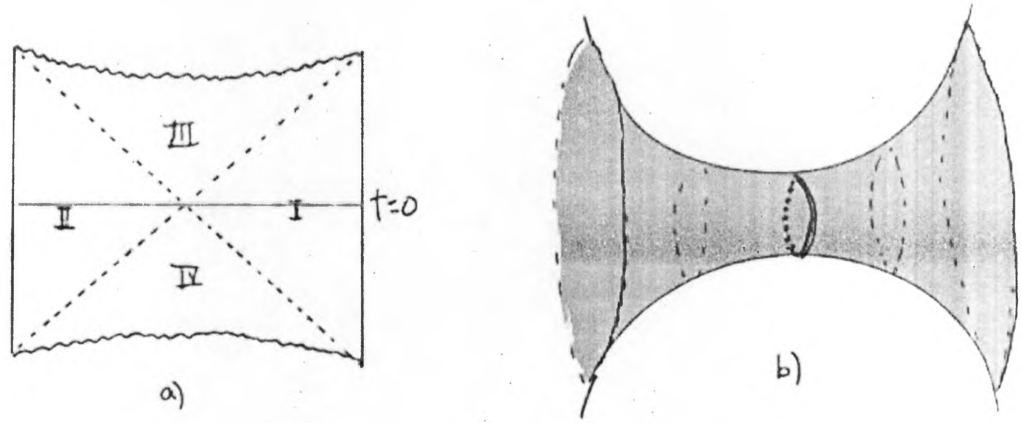

Figure 5.1: a) Penrose diagram for the spacetime of the maximally extended two sided AdS-Schwarzschild black hole, regions I and II are exterior regions and III and IV are exterior regions b) Spacial geometry of $t = 0$ slice (shown in red), showing the horizon (dashed). Image from [14]

is described by a thermal state of the CFT on a sphere, and the area of the black hole can be identified with the entropy of the CFT. The entropy of entanglement $S = -tr(\rho_i \log \rho_i)$ is the Beckenstein-Hawking entropy is given by the area of the event horizon, $S = A/4G$.

Maldacena in [6] argued that the the spacetime associated with the thermofied double state (equation 5.10) of a two CFT system is maximally extended AdS-Schwarzschild black hole (see the figure 5.1) . The geometry has two asymptotic regions, each with its own boundary and black hole exterior.

The two individual states states in the thermofield double are product of states

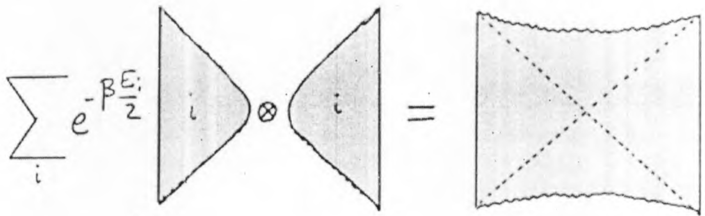

Figure 5.2: Gravity interpretations for the thermofield double state in a quantum system defined by two non interacting CFTs.[14]

in a non-interacting CFT. For each state of the CFT the corresponding geometry has nothing to do with the other, and so must correspond two complete separate AdS spacetimes. On the other hand the state described in equation (5.10) corresponds to an extended black hole where the two sides are connected by classical spacetime in the form of a wormhole. We can make a remarkable conclusion from this: By entangling the degrees of freedom in two separate gravity theories in a particular way, we can glue the corresponding geometries. In the thermofield double state, the black hole entropy is the entropy of a single CFT or the entanglement entropy of the two subsystems with each other.

5.3 Ryu-Takayanagi formula

The Beckenstein-Hawking formula gives the total entropy of a CFT in a thermal state, identifying it by the area of the horizon in the dual spacetime. Ryu and

Takayanagi proposed a formula [10] for the entropy of any spacial subsystem of a CFT associated with some classical spacetime.

Consider a CFT with a holographic dual defined on some spacetime geometry $\mathcal{B}$. Now let the CFT be in the state $|\Phi\rangle$ and the associated dual geometry G_Φ. Now we can consider a subsystem A in the CFT by choosing a spacial slice $\Sigma_\mathcal{B}$ of $\mathcal{B}$ and choosing $A \subset \Sigma_\mathcal{B}$ of this slice. Since the boundary geometry ∂M_Φ of M_Φ is the same as $\mathcal{B}$, we can define regions on ∂M_Φ corresponding to $\Sigma_\mathcal{B}$, A and $\bar{A} \subset \Sigma_\mathcal{B}$.

Let S_A be the entanglement entropy of the subsystem A. The Ryu-Takayanagi formula states that this entropy is equal to the area of a certain codimension-2 surface $\tilde{A}$ in M_Φ that is homologous to A

$$S_A = \frac{1}{4G} Area(\tilde{A}) \tag{5.11}$$

The surface is defined in figure (5.3) and satisfies the following conditions.

- The surface $\tilde{A}$ has the same boundary as the A.
- The surface $\tilde{A}$ is homologous to the surface A. Surfaces $\tilde{A}$ and A can fail to be homologous if the bulk geometry is a black hole.
- The surface $\tilde{A}$ extremizes the area, so if there are multiple surfaces that are possible then $\tilde{A}$ is the one with the least area.

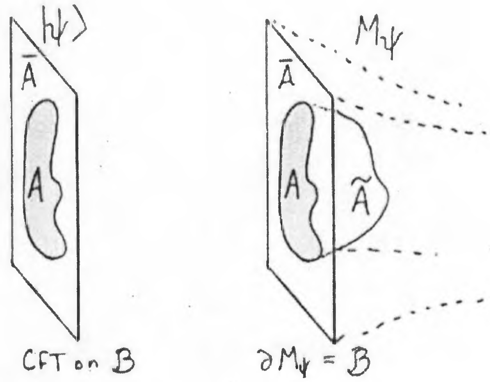

Figure 5.3: In this diagram the time direction is suppressed. The left side shows the spacial slice $\Sigma_{\mathcal{B}}$ on which the CFT lives. The right side shows the geometry M_{Φ} dual to the state $|\Phi\rangle$. Image taken from [14]

Example

Consider the entanglement entropy of a ball-shaped region B for a CFT in the vacuum state on $R^{d-1,1}$. The dual geometry to this is the Poincare AdS

$$ds^2 = \frac{l^2}{z^2}(-dt^2 + (dx^i)^2 + dz^2) \tag{5.12}$$

Where l is the AdS radius. We need to find the extremal area $d-1$ dimensional surface whose geometry on the boundary is same as the ball B, which we choose to be at $(x^i)^2 = R^2$ and $t = 0$. The AdS geometry is static, so the bulk extremal surface should lie in the $t = 0$ slice.

We can parametrize the surface as $X^\mu(\sigma)$. In $d-1$ dimensions the area functional is given by

$$Area = \int d^{d-1}\sigma \sqrt{\det g_{ab}} \tag{5.13}$$

Where g_{ab} is the induced metric on the surface given by

$$g_{ab} = G_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^a}\frac{\partial X^\nu}{\partial \sigma^b} \tag{5.14}$$

Here $G_{\mu\nu}$ is the metric of spacetime. We can now take σ to be the coordinates x^i, with the surface parametrized by $z(x^i)$ and $t(x^i) = 0$. We have

$$\begin{aligned} g_{ij} &= \frac{l^2}{z^2}\left(\frac{\partial z}{\partial x^i}\frac{\partial z}{\partial x^j} + \frac{\partial x^k}{\partial x^i}\frac{\partial x_k}{\partial x^j}\right) \\ &= \frac{l^2}{z^2}\left(\frac{\partial z}{\partial x^i}\frac{\partial z}{\partial x^j} + \delta_{ij}\right) \end{aligned} \tag{5.15}$$

For simplicity lets take $d = 2$ so $x^i = x$ and g_{ij} is just a 1x1 matrix, and

$$\det g = \frac{l^2}{z^2}(1 + \left(\frac{\partial z}{\partial x}\right)^2) \tag{5.16}$$

So the area functional becomes

$$Area = \int dx \frac{l}{z}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2} \tag{5.17}$$

We can now use Euler-Lagrange equations to extremize the area. Let's call $\frac{\partial z}{\partial x} = \dot{z}$.

Thus we have

$$\frac{d}{dx}\left(\frac{\partial}{\partial \dot{z}}\frac{l}{z}\sqrt{1+\dot{z}^2}\right) = \frac{\partial}{\partial z}\frac{l}{z}\sqrt{1+\dot{z}^2} \tag{5.18}$$

Solving this equation and setting $z(-R) = 0$ and $z(R) = 0$ we get $z^2 + x^2 = R^2$. So the extremal area is

$$Area = \int dx \frac{l}{z}\sqrt{1+\frac{x^2}{z^2}} \tag{5.19}$$

Note that this area diverges as $z = 0$ is a singular point, so we regularize the integral by introducing a cutoff ϵ and integrate over all regions where $z > \epsilon$. Thus,

$$\begin{aligned} Area &= \int_{-\sqrt{R^2-\epsilon^2}}^{\sqrt{R^2-\epsilon^2}} dx \frac{l}{\sqrt{R^2-x^2}}\sqrt{1+\frac{x^2}{R^2-z^2}} \\ &= 2l \log \frac{2R}{\epsilon} \end{aligned} \tag{5.20}$$

So the entropy is

$$S = \frac{l}{2G}\log\frac{L}{\epsilon} \tag{5.21}$$

Where $L = 2R$ is the length of the interval. Now, the entanglement entropy of a CFT in the vacuum state with central charge c in the interval L in terms of UV cutoff is given by

$$S = \frac{c}{3}\log\frac{L}{\epsilon} \tag{5.22}$$

This precisely agrees with our calculation for a CFT with central charge $c = 3l/2G$.

5.4 Mutual Information

The Ryu-Takayanagi formula has a problem. The area A is divergent as there is infinite proper distance on the boundary of AdS. To make sense of this formula we have various options, firstly we can work with a UV cutoff at some high scale $1/\epsilon$. In the geometry part we can keep only the $z > \epsilon$ part as done above. We can work with this usual approach of quantum field theory and work with quantities that remain finite when the cutoff is removed. We can define a quantity called mutual information $I(A:B)$ by

$$I(A:B) = S(A) + S(B) + S(A \cup B) \tag{5.23}$$

We can use this to obtain a regulated version of the entropy by choosing B to be all the points within a distance $\leq \epsilon$ from A.

Mutual information is a measure of entanglement and correlations between two subsystems A and B. Mutual information provides an upper bound for all correlations between two subsystems; if $\mathcal{O}_A$ and $\mathcal{O}_B$ are two bounded operators acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, it can be shown that[14]

$$\frac{(\langle\mathcal{O}_A\mathcal{O}_B\rangle) - \langle\mathcal{O}_A\rangle\langle\mathcal{O}_B\rangle)^2}{2|\langle\mathcal{O}_A\rangle|^2|\langle\mathcal{O}_B\rangle|^2} \leq I(A;B) \tag{5.24}$$

Let's calculate the mutual information for the CFT discussed before. Consider the mutual information of a CFT in the vacuum state between intervals A and B

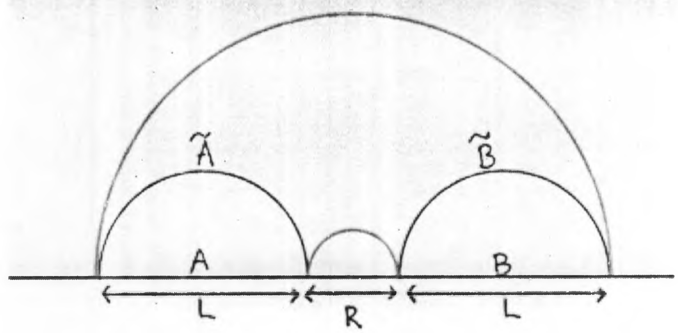

Figure 5.4: Extremal surfaces for calculating the mutual information between two disjoint regions in a two-dimensional CFT.

with length L separated by a distance R(see figure). The mutual information is $I(A:B) = S(A) + S(B) - S(A \cup B)$.

For this example there are two extremal surfaces of $A \cup B$.

First one is the union of surfaces $\tilde{A}$ and $\tilde{B}$. The area of the extremal surfaces is

$$A_1 = 4l \log \frac{L}{\epsilon} \tag{5.25}$$

So the mutual information is zero.

The second one is the union of the red surfaces. In this case the area of the extremal surfaces is

$$A_2 = 2l \left(\log \frac{2L+R}{\epsilon} + \log \frac{R}{\epsilon} \right) \tag{5.26}$$

So the mutual information is

$$I(A:B) = \frac{l}{2G} \log \frac{R(R+2L)}{L^2} \tag{5.27}$$

The surface we choose from these two should be the one with least area. So we choose A_1 for $R > (\sqrt{2}-1)L$ and we choose A_2 for $R < (\sqrt{2}-1)$. Thus we can now state that the mutual information is always positive and it's extremal value is zero.

5.5 Chaos

Thermal systems also have another basic property. Starting from a state $|\Phi\rangle$ the system evolves into a much disordered state than the original state. These final states depend very sensitively on the initial states, seemingly similar initial states could evolve into final states that are quite different from the initial states. Such chaotic behavior has come to be referred to as "scrambling", and it has been conjectured that black holes are the fastest scramblers in nature. The time it takes for fast scramblers to render the density matrix for a subsystem A to exactly thermal is conjectured to be $t \sim \beta \log S$, where S is the entropy of the system. We will now discuss the interplay of entanglement and scrambling using holographic tools as discussed in[12].

Here we discuss the eternal black hole setup as above with regions A in the L system and region B in the R system. We will see what happens when we introduce a small initial perturbation. We choose regions A and B in L and R CFTs at time $t = 0$. Here A and B may be highly entangled. We now consider the effect of introducing a small amount of energy E into the L system at time $t = -t_w$, by throwing a few quanta in towards the horizon. One expects that the CFTs dual to the geometry will have sensitive dependence on initial conditions and a small perturbation should show chaotic behavior. Thus at $t = 0$ we expect to not be in the thermofield double state and should have less entanglement between A and B.

Naively one might think that introduction of a few quanta should not have an effect on the geometry, however the time $t = 0$ defines a frame in the bulk and relative to this frame, the energy introduced at time t_w in the past will be blue-shifted. So the addition of quanta at time $t = -t_w$ will affect entanglement. The related geometry can be described by shock wave [4].

5.6 Black holes and butterfly effect

In this section we will discuss the geometrical constructions used to calculate the mutual information holographically, assuming Einstein gravity as done in [12]. We will see a bulk geometry that illustrates the sensitivity of specific entanglements in the thermofield double state to mild perturbations in the past. We will work with Einstein gravity in 2+1 bulk dimensions. We will use RT surfaces and correlation function probes to follow the loss of correlation between L and R sides.

5.6.1 Unperturbed BTZ black hole

Let's consider geometrical dual of the unperturbed thermofield double state (equation 5.10). In 2+1 bulk dimensions the thermofield double state corresponds to the BTZ metric. We can think of the CFTs to live on the boundaries of the respective regions. The BTZ metric is given by

$$ds^2 = -\frac{r^2 - R^2}{l^2} dt^2 + \frac{l^2}{r^2 - R^2} dr^2 + r^2 d\phi^2 \tag{5.28}$$

where l is the AdS radius, and R is the horizon radius given by $R^2 = 8GMl^2$. The temperature is given by $\beta = 2\pi l^2/R$. As done earlier, it is useful to switch to Kruskal coordinates. In these coordinates the metric is

$$ds^2 = \frac{-4l^2 dudv + R^2(1-uv)^2 d\phi^2}{(1+uv)^2} \tag{5.29}$$

The boundaries are at $uv = -1$ and the singularities are at $uv = 1$.

In our discussion it would be required to compute the geodesic distances between points in the BTZ geometry. Since BTZ is a quotient of AdS, we can use the formula for geodesic distance in pure AdS_{2+1} given by [?]

$$\cosh \frac{d}{l} = T_1 T_1' + T_2 T_2' - X_1 X_1' - X_2 X_2' \tag{5.30}$$

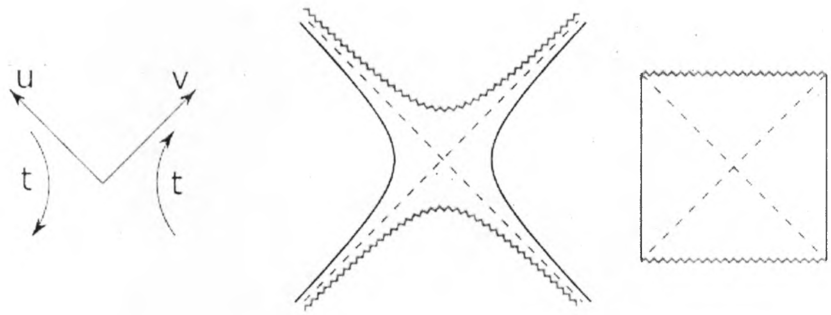

Figure 5.5: Kruskal diagram and Penrose diagram for unperturbed BTZ

we have used the embedding coordinates

$$\begin{aligned} T_1 &= \frac{u+v}{1+uv} = \frac{1}{R}\sqrt{r^2-R^2}\sinh\frac{Rt}{l^2} \\ T_2 &= \frac{1-uv}{1+uv}\cosh\frac{R\phi}{l} = \frac{r}{R}\cosh\frac{R\phi}{l} \\ X_1 &= \frac{v-u}{1+uv} = \frac{1}{R}\sqrt{r^2-R^2}\cosh\frac{Rt}{l^2} \\ X_2 &= \frac{1-uv}{1+uv}\sinh\frac{R\phi}{l} = \frac{r}{R}\sinh\frac{R\phi}{l} \end{aligned} \tag{5.31}$$

This also gives a relation to go from u, v to r, t.

5.6.2 BTZ shock wave

In this section we will mildly perturb the BTZ metric by adding a few particles at the left boundary and let them fall into the black hole. If we release a perturbation with energy E in the past at time $-t_w$ then it will cross the $t = 0$ time slice with energy

$$E_p \sim \frac{El}{R} e^{Rt_w/l^2} \tag{5.32}$$

This geometry is constructed by gluing a BTZ solution of mass M to a solution of mass $M + E$ across the null surface $u_w = e^{-Rt_w/l^2}$. Here E is the asymptotic energy of the perturbation, which is taken to be small compared to M.

We take coordinates u, v to the right (past) of the shell and $\tilde{u}, \tilde{v}$ to the left (future), so the metric is always in the form of equation (5.25). Because we have an increase in mass in the left we also have an increase in the radius to the left. If R is the radius to the right then $\tilde{R} = \sqrt{\frac{M+E}{M}} R$.

...

For small E/M, the solution is a simple shift,

$$\tilde{v} = v + \alpha, \quad \alpha \equiv \frac{E}{4M} e^{Rt_w/l^2} \tag{5.33}$$

This is exact if we take $E/M \to 0$ and $t_w \to \infty$ with α fixed. In this limit, the metric can be rewritten as

$$ds^2 = \frac{-4l^2 du dv + R^2[1 - u(v + \alpha\theta(u))]^2 d\phi^2}{[1 + u(v + \alpha\theta(u))]^2} \tag{5.34}$$

The geometry that corresponds the this metric is shown the figure..... It is sometimes useful for computations to use discontinuous coordinates $U = u, V = v + \alpha\theta(u)$. In these coordinates the metric takes the form

$$ds^2 = \frac{-4l^2 dUdV + 4l^2\delta(U)dU^2 R^2(1-UV)^2 d\phi^2}{(1+UV)^2} \tag{5.35}$$

Equation (5.34) satisfies Einstein equations for the stress tensor

$$T_{uu} = \frac{\alpha}{4\pi H}\delta(u) \tag{5.36}$$

5.6.3 Geodesic distance

Consider a geodesic connecting a point t_L on the left boundary with a point t_R on the right boundary, where t_L and t_R are killing times located at the same ϕ. A geodesic connecting these points will pass through $u = 0$ at some value of v. We can now use formula for geodesic distance (equation 5.29) to compute d_1 from the left boundary to this point and d_2 from the right boundary to this point. We get

$$\begin{aligned} \cosh\frac{d_1}{l} &= \frac{r}{R} + \frac{1}{R}\sqrt{r^2 - R^2} e^{-Rt_L/L^2}(v+\alpha) \\ \cosh\frac{d_2}{l} &= \frac{r}{R} - \frac{1}{R}\sqrt{r^2 - R^2} e^{Rt_R/l^2} v \end{aligned} \tag{5.37}$$

We can extremize $d_1 + d_2$ over v to find the total geodesic distance d. For large r we get

$$\frac{d}{l} = 2\log\frac{2r}{R} + 2\log\left(\cosh\frac{R}{2l^2}(t_R - t_L) + \frac{\alpha}{2}e^{-r(t_L+t_R)/2l^2}\right) \tag{5.38}$$

Now $\alpha = 0$ represents the unperturbed BTZ, so the contribution of α represents an increase in distance due to the shock wave.

If $t_L + t_R$ is sufficiently large then the shockwave does not affect the geodesic distance.

We can also calculate the distance between equal time points on the same boundary with an angular separation ϕ, this distance is unaffected by the shockwave and is given by

$$\frac{d}{l} = 2\log\frac{2r}{R} + 2\log\sinh\frac{R\phi}{2l} \tag{5.39}$$

5.6.4 Mutual information

We have constructed a perturbed thermofield double state. We can now use the above calculations to understand the measure of correlations between regions $A \subset L$ and $B \subset R$ in the two CFT's. One of these is the mutual information $I(A:B) = S_A + A_B - S_{A\cup B}$. We can use the Ryu-Takayanagi formula (Equation 5.11) to compute the entropy S_Ω of the density matrix associated to the region Ω. In 2+1 dimensional bulk, the extremal surfaces are geodesics, and the area is the length of the geodesics.

We will consider a spacial region at $t = 0$ consisting of two disconnected components A and B. For simplicity, we will take $\phi < \pi$, and centre then at the same angular location.

Let's first consider S_A. Here we have two choices of extremal surfaces. First is the geodesic that connects the endpoints on interval A, this is calculated using

Equation (5.38). The second surface is a geodesic that connects one endpoint to the image of the other by the BTZ identification, plus a contribution from the horizon of the black hole as required by the homology condition. When $\phi < \pi$, the first surface always has smaller area, so we can use equation (5.38) to get

$$S_A = S_B = \frac{l}{4G}\left(2\log\frac{2r}{R} + 2\log\sinh\frac{R\phi}{2l}\right) \tag{5.40}$$

We can now calculate $S_{A\cup B}$. We have taken $\phi < \pi$, so we again have two choices of extremal surfaces. The first one is the sum of the two geodesics used to compute S_A and S_B. This gives $S^1_{A\cup B} = S_A + S_B$. The second choice we have is the geodesic connecting the endpoints of A to the endpoints of B. We can use Equation (5.37) for this purpose. Thus we obtain

$$S^2_{A\cup B} = \frac{l}{G}\left[\log\frac{2r}{R} + \log\left(1+\frac{\alpha}{2}\right)\right]. \tag{5.41}$$

The Ryu Takalanagi formula tells us that we should use the lowest of these two. For regions with $\sinh\frac{R\phi}{2l} < 1$ we have $S^1_{A\cup B} < S^2_{A\cup B}$, giving us $I(A:B) = 0$. For larger regions with sufficiently small α, $S^1_{A\cup B} > S^2_{A\cup B}$, and we have positive mutual information. Substituting for α using equation (5.32) and rewriting M and R in terms of Bekensiein-Hawking entropy S and inverse temperature β we get

$$I(A;B) = \frac{l}{G}\left[\log\sinh\frac{\pi\phi l}{\beta} - \log\left(1+\frac{E\beta}{4S}e^{2\pi t_w/\beta}\right)\right] \tag{5.42}$$

We can see that the mutual information decreases as t_w increases. For high temperature, I reaches zero when t_w is

$$t_*(\phi) = \frac{\phi l}{2} + \frac{\beta}{2\pi} \log \frac{2S}{\beta E} \tag{5.43}$$

In large N gauge theory, string coupling g_s is small, so $S \sim N^2$, and E assumes its smallest reasonable value $E \sim T = 1/\beta$ then

$$t_* = \frac{\beta}{2\pi} \log S \tag{5.44}$$

We can see that I evolves with a sharp transition in which the $A \cup B$ minimal surfaces exchanges dominance and eventually, I goes to zero in a continuous but non-differentiable way. Here t_* is the scrambling time.

Thermodynamic Quantities of the BTZ black hole

Consider the action for gravity in (2+1) dimensions

$$S = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g}(R - 2\Lambda) \tag{5.45}$$

When the cosmological constant is zero, the vacuum solution is necessarily flat space. But in 1992 Banados, Teitelboim and Zanelli discovered the BTZ black hole solution [1] with a negative cosmological constant.

With a negative cosmological constant $\Lambda = -1/l^2$, the BTZ solution to (2+1) dimensional gravity is the metric

$$ds^2_{BTZ} = -V(\rho)dt^2 + V(\rho)^{-1}d\rho^2 + \rho^2 \left(d\phi + \frac{4G_3 J}{\rho^2}dt\right)^2, \tag{5.46}$$

$$V(\rho) = \left(-8G_3 M + \frac{\rho^2}{l^2} + \frac{16G_3^2 J^2}{\rho^2}\right), \tag{5.47}$$

where ϕ is periodic with period 2π. Notice that $V = 0$ at two values of ρ. Thus the black hole has two event horizons (boundaries) $\rho_\pm$ where,

$$\rho_\pm = 2\left(G_3 l(lM \pm \sqrt{l^2M^2 - J^2})\right)^{1/2} \tag{5.48}$$

The mass and angular momentum of the black hole is given by

$$\begin{aligned} M &= \frac{\rho_+^2 + \rho_-^2}{8l^2G_3} \\ J &= \frac{\rho_+\rho_-}{4LG_3} \end{aligned} \tag{5.49}$$

Substituting $J = 0$ and $M = 0$ gives us the AdS_3 black hole in local coordinates.

We can now calculate the temperature and the entropy as discussed earlier in Chapter 2.

We can now write down the action for the BTZ black hole

$$S_{total} = S_{Bulk} + S_{boundary} + S_{CT} \tag{5.50}$$

where the $S_{boundary}$ is the boundary term also called the Gibbons-Hawking boundary term that needs to be added to the Einstein-Hilbert action when the spacetime manifold has a boundary and S_{CT} is the counter term that is necessary to get rid of divergences.

now,

$$\begin{aligned} S_{bulk} &= \frac{1}{16\pi G}\int \sqrt{-g}\left(R+\frac{2}{l^2}\right) d^3x \\ &= \frac{2\pi\beta}{16\pi G}\int_{\rho_+}^{\lambda} \sqrt{-g}\left(R+\frac{2}{l^2}\right) d\rho \\ &= \frac{\beta(\rho_+^2-\lambda^2)}{4Gl^2} \end{aligned} \tag{5.51}$$

In the end of the calculation we must take λ to infinity as we are integrating over all space.

The Gibbons-Hawking term on the boundary is given by [5]

$$\begin{aligned} S_{boundary} &= \frac{2}{16\pi G}\int K d^2x \\ &= \frac{2\pi\beta}{8\pi G} K|_{\rho=\lambda} \end{aligned} \tag{5.52}$$

where K is given by

$$K = n^{\mu}\frac{\partial\sqrt{h}}{\partial x^{\mu}} \tag{5.53}$$

here $\sqrt{h}$ is the determinant of the metric on the boundary. Thus we have

$$S_{boundary} = \frac{\beta(-8Gl^2M+\rho_+^2+\lambda^2)}{4Gl^2} \tag{5.54}$$

and

$$\begin{aligned} S_{CT} &= \frac{-\beta}{4Gl}\sqrt{h}|_{\rho=\lambda} \\ &= -\frac{\beta}{4Gl}\sqrt{\frac{(\rho_+^2-\lambda^2)(\rho_-^2-\lambda^2)}{l^2}} \end{aligned} \tag{5.55}$$

Adding everything we get

$$S_{total} = \frac{\beta}{4Gl^2}\left(\rho_-^2 - \lambda^2 + \sqrt{(\rho_+^2-\lambda^2)(\rho_-^2-\lambda^2)}\right) \tag{5.56}$$

Now formally taking the limit $\lambda \to \infty$ we get

$$S_{total} = \frac{(\rho_+^2 - \rho_-^2)\beta}{8Gl^2} \tag{5.57}$$

Now that we have the total action we can use concepts discussed in chapter 3 to calculate thermodynamic quantities.

Let's define a tangent vector field to the euclidean time circle that shrinks to zero at the horizon

$$\partial_i = \partial_t - \omega\partial\phi = T_\mu dx^\mu \tag{5.58}$$

Where ω is conjugate to angular momentum J. W can find the value of ω by requiring the tangent vector to go to zero at the horizon. The tangent vector P is given by

$$P = T_\mu G^{\mu\nu} T_\nu \tag{5.59}$$

Where $G^{\mu\nu}$ is the metric tensor. Setting $P = 0$ at $\rho = \rho_+$ we get

$$\omega = -\frac{\rho_-}{l\rho_+} \tag{5.60}$$

The temperature can be calculated by requiring the period of the euclidean time to be 2π. Doing this we get

$$T = \frac{\rho_+ - \rho_-}{2\pi\rho_+} \tag{5.61}$$

The free energy is defined as

$$\begin{aligned} F &= -\frac{S_{total}}{\beta} \\ &= -\frac{l^2\pi^2T^2}{2G - 2Gl^2\omega} \end{aligned} \tag{5.62}$$

The entropy is given by

$$S = -\frac{\partial F}{\partial T} = \frac{\pi\rho_+}{2G} \tag{5.63}$$

The area of the horizon is $2\pi\rho_+$, so we can say that the entropy and area are related by

$$S = \frac{Area}{4G} \tag{5.64}$$

Bibliography

[1] Maximo Banados, Claudio Teitelboim, and Jorge Zanelli, *The Black hole in three-dimensional space-time*, Phys. Rev. Lett. **69** (1992), 1849–1851.

[2] J. M. Bardeen, B. Carter, and S. W. Hawking, *The four laws of black hole mechanics*, Communications in Mathematical Physics **31** (1973), 161–170.

[3] Jacob D. Bekenstein, *Black hole hair: 25 - years after*, Physics. Proceedings, 2nd International A.D. Sakharov Conference, Moscow, Russia, May 20-24, 1996, 1996, pp. 216–219.

[4] P. A.M. Dirac, *General theory of relativity*, Princeton University Press, 1996.

[5] Tevian Dray and Gerard 't Hooft, *The effect of spherical shells of matter on the schwarzschild black hole*, Comm. Math. Phys. **99** (1985), no. 4, 613–625.

[6] G. W. Gibbons and S. W. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, Phys. Rev. **D15** (1977), 2752–2756.

[7] Juan Maldacena, *Eternal black holes in anti-de Sitter*, JHEP **04** (2003), 021.

[8] Makoto Natsuume, *AdS/CFT Duality User Guide*, Lect. Notes Phys. **903** (2015), pp.1–294.

[9] Joseph Polchinski, *String theory*, Cambridge University Press, Cambridge, 2005.

[10] Shinsei Ryu and Tadashi Takayanagi, *Aspects of Holographic Entanglement Entropy*, JHEP **08** (2006), 045.

[11] J. J. Sakurai, *Modern quantum mechanics*, Dorling Kindersley Pearson Education, New Delhi, India, 2014.

[12] Stephen H. Shenker and Douglas Stanford, *Black holes and the butterfly effect*, JHEP **03** (2014), 067.

[13] Leonard Susskind, *The World as a hologram*, J. Math. Phys. **36** (1995), 6377–6396.

[14] Mark Van Raamsdonk, *Lectures on Gravity and Entanglement*, Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015, 2017, pp. 297–351.

[15] Steven Weinberg, *Gravitation and cosmology: Principles and applications of the general theory of relativity*, John Wiley & Sons, Inc., 1972.

[16] Barton Zwiebach, *A first course in string theory*, Cambridge University Press, Cambridge New York, 2009.